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ON THE GROUPS GENERATED BY TWO OPERATORS OF ORDER THREE WHOSE PRODUCT IS ALSO OF ORDER THREE.

BY G. A. MILLER.

Two operators of order two always generate a dihedral rotation group whose order is twice the order of their product.[*] In this case, the group generated by two operators is completely defined by the orders of the operators and that of their product. Similarly, the group generated by two operators of order three, whose product is of order two, is completely defined by the orders of the two generating operators and that of their product. That is, the alternating group of order 12 is the only one which satisfies these conditions.[†] On the contrary, two operators (s_1, s_2) of order three whose product is also of this order may generate any one of an infinite system of groups of a finite order. The present paper is devoted to the determination of some of the properties of the groups of this system.[‡]

The following conjugate commutators

$$s_3 = s_1^2 s_2^2 s_1 s_2, \qquad s_4 = s_2^2 s_1^2 s_2^2 s_1 s_2^2 = s_1 s_2 s_1^2 s_2^2, \qquad s_5 = s_2 s_1^2 s_2^2 s_1$$

of such a group (G)[§] are commutative since $s_3 s_4 s_5 = s_4 s_3 s_5 = 1$. The last equation may readily be verified by means of the equations $(s_1 s_2)^3 = s_1 s_2 s_1 s_2 s_1 s_2 = 1$ or $s_1 s_2 s_1 = s_2^2 s_1^2 s_2^2$. Hence s_3, s_4, s_5 generate an abelian subgroup (H) of G which is either cyclic or may be generated by two independent operators. We shall soon see that H is the commutator subgroup of G. From the equations

$$s_1^2 s_3 s_1 = s_1 s_2^2 s_1 s_2 s_1 = s_4, \qquad s_1^2 s_4 s_1 = s_2 s_1^2 s_2^2 s_1 = s_5, \qquad s_1^2 s_5 s_1 = s_3,$$

it follows that H is invariant in G. Its order (h) is the order (g) of G divided by either 3 or 9, since the group generated by s_2 and H contains $s_1^2 s_2^2 s_1 = s_2^2 s_5$ and hence is also invariant in G. This proves that H is the commutator subgroup of G.* Moreover, the given equations show that s_1 and s_2 transform the operators s_3, s_4, s_5 in the same manner. Having determined some of the properties which all of the groups under consideration possess, we proceed to determine some of the special properties which depend upon the factors of h and especially to determine the necessary and sufficient conditions that such groups exist.

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[*] *Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

[†] *Cf.* Burnside, *Theory of Groups of Finite Order*, 1897, p. 291.

[‡] *Cf. ibid.*, p. 296.

[§] In what follows the symbol G will always represent a group of the system under consideration.

When H is cyclic and of order p^a, p being any prime number, the matter is quite simple. If $p \equiv 2 \bmod 3$, it follows from the above equations that G is abelian.† In this case $s_3 s_4 s_5 = s_3^3 = 1$ and therefore $h = 1$. The non-cyclic group of order 9 is clearly the only group which satisfies these conditions. It is the only abelian group in the infinite system of groups under consideration and will not be considered in what follows. If $p \equiv 1 \bmod 3$, s_2 and H generate one and only one group of this system for every value of $a > 0$. Its order is $3h$ and it contains $2h$ operators of order 3. The direct product of any one of these groups and an operator of order 3 is of order $9h$ and belongs to this system.

That there is only one group of order $9h$ for every value of a may be proved as follows: Such a G is isomorphic to the group of order 3 with respect to those of its operators which are commutative with each operator of H. That is, G contains a cyclic invariant subgroup of order $3h$ and hence an invariant subgroup of order 3. It also contains an invariant subgroup of order $3h$, generated by H and s_2, which does not include any invariant group of order 3. In other words, it is the direct product mentioned above. Hence *there are two and only two groups for every value of* $a > 0$ *whenever* $p \equiv 1 \bmod 3$. When $p = 3$, it follows from $s_3 s_4 s_5 = 1$ that H is either the identity or of order 3.‡ In this case there is therefore only one group; viz. the non-abelian group of order 27 which includes no operator of order 9.

From what precedes, it follows that we can construct at least one group of the required system whenever H is cyclic and all the prime factors of h are $\equiv 1 \bmod 3$, or when h is three times such a number. Moreover, it is not difficult to see that the equation $s_3 s_4 s_5 = 1$ cannot be satisfied under any other conditions, since H is the direct product of its cyclic subgroups, each of whose orders is a power of a single prime. It remains to consider the cases when

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* *Quarterly Jour. of Math.*, vol. 28, 1896, p. 266; *Cf.* Weber, *Algebra*, vol. 2, 1899, p. 131.

† The group of isomorphisms of the cyclic group of order p^a is of order $p^{a-1}(p-1)$.

‡ Burnside, *l. c.*, p. 76.

H is non-cyclic, but is generated by two of its operators. We shall again begin with the case when $h = p^\alpha$.

When $p \equiv 2 \bmod 3$, s_3 and s_4 must be the independent generators of H, since the cube of a common operator is identity. Similarly, we observe that s_3 and s_4 cannot have more than 3 common operators whenever $p = 3$. When $p \equiv 1 \bmod 3$, we may construct a G by establishing a (p^α, p^δ) isomorphism between two G's of orders $3p^\alpha$, $3p^\delta$ respectively in such a manner that the resulting group contains subgroups of order p which are not invariant. Hence, in this case, the two invariants of H may have any orders such that their product is equal to h; *i. e.* s_3 and s_4 may have a common subgroup of any order less than the order of s_3.

It has been observed that the two invariants of H are equal when $p \equiv 2 \bmod 3$. It may therefore be assumed that the order of s_3 and s_4 is p^m, where $m = \frac{a}{2}$, and that H is of type (m, m). H contains $p^{m-1}(p + 1)$ cyclic subgroups of order p^m, which may be divided into $p + 1$ sets such that each one of those of one set has only identity in common with any one of another set. These subgroups are transformed by the holomorph of H according to a transitive substitution group of degree $p^{m-1}(p + 1)$ and of order $p^{3m-2}(p^2 - 1)$. Since $p \equiv 2 \bmod 3$, the group of isomorphisms of H contains operators of order 3 which permute each of its cyclic subgroups of order p^m.

Such an operator (s_1) transforms the $p + 1$ sets mentioned above as units. It cannot transform any one of these sets into itself, since such a set is composed of p^{m-1} groups of order p^m. Hence H and s_1 generate a group which contains $2h$ operators of order 3. For s_2 we may select any substitution which transforms the operators of H in the same manner as s_1 does and which gives a product of order p^m when multiplied by s_1. This proves the existence of the groups in question for every value of $p \equiv 2 \bmod 3$ and for every value of $m > 0$.

When $p = 3$ and the two independent generators (t_1, t_2) of H are either of order 3^m or of orders 3^m and 3^{m-1} respectively, it is easy to verify that there is an operator of order 3 in the group of isomorphisms of H, which transforms t_1 into $t_2 t_1$ and t_2 into $t_1^{-3} t_2^{-2}$. This operator, s_1, and H generate a group of order $3h$ which contains $2h$ operators of order 3 that are not found in H; for, $(t_1^\alpha t_2^\beta s_1)^3 = t_1^\alpha t_2^\beta t_2^{-\alpha} t_1^{-2\alpha} t_2^\beta t_1^{3\beta} t_2^\alpha t_1^\alpha t_1^{-3\beta} t_2^{-2\beta} = 1$. If we take for s_2 an operator such that $s_1^2 s_2 = t_1$, it is evident that s_1, s_2 generate G. This proves the existence of groups whenever $p = 3$ and H has two invariants of order 3^m or one invariant of order 3^m and the other of order 3^{m-1}. It was proved above that these conditions are also necessary. It remains, therefore, only to examine the case when H has two independent generators whose orders are not powers of the same prime number.

In this general case, we may write h in the form $3^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \ldots$ and observe that each of the subgroups whose orders are $3^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots$ is either cyclic or the direct product of two cyclic groups.* The necessary and sufficient conditions that s_1 and s_2 can be so chosen as to generate a group which contains any one of these subgroups, as H, have been determined. If these conditions are satisfied for each of the given subgroups, we may suppose s_1 and s_2 so formed as to generate the group formed by establishing a $(3^{\alpha_0}, p_1^{\alpha_1}, p_2^{\alpha_2} \ldots)$ isomorphism between the separate group whose h's are powers of primes. If they are not satisfied in each instance, it follows from the equation $s_3 s_4 s_5 = 1$ that there is no group in the infinite system under consideration which has this H. Hence this general case is included under the special cases considered above.

CORNELL UNIVERSITY, JULY, 1901.

ON THE INVARIANTS OF A QUADRANGLE UNDER THE LARGEST SUBGROUP, HAVING A FIXED POINT, OF THE GENERAL PROJECTIVE GROUP IN THE PLANE.**

BY W. A. GRANVILLE.

IN the ANNALS OF MATHEMATICS, vol. 12, p. 82, Professor Lovett proposes the problem of finding the invariants of a quadrangle under the transformations of the six parameter group in the plane generated by the infinitesimal transformations:

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

$$p = \frac{\partial f}{dx}, \qquad q = \frac{\partial f}{dy}.$$

This is the largest subgroup of the general projective group in the plane, which has a fixed point. The invariants found by Professor Lovett were

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* *Cf. Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

** Presented to the American Mathematical Society at its meeting, April 27, 1901.